Chapter 2: Cardinality

Owen Wang

Exercise 2.4. ⋆

(a) Let f be the identity function. Then, f is a bijection from $A \to A$, thus $A \sim A$.

(b) $A \sim B$ implies that there exists $f : A \to B$ that is a bijection. Thus, for $a \in A$ and $b \in B$, there exists a unique a such that $f(a) = b$. Then, define an inverse g such that it maps b to the unique a.

g is injective: Assume $g(x) = g(y)$.

$$\begin{aligned} g(x) &= g(y) \\ f(g(x)) &= f(g(y)) \\ x &= y \end{aligned}$$

g is surjective: Take any arbitrary $a \in A$. Then, let $b = f(a)$. Because f is surjective, a exists for every b.Then, apply g such that $g(a) = b$. Then, for every b, there exists a a such that $g(a) = b$.

(c) Assume that $f : A \to B$ and $g : B \to C$ are bijections. Then, we aim to show that $h = (g \circ f) : A \to C$ is a bijection.

As f and g are both bijections, f^{-1} and g^{-1} exist. Then, assume that $h(x) = h(y)$. Then,

$$\begin{aligned} h(x) &= h(y) \\ g(f(x)) &= g(f(y)) \\ f(x) &= f(y) \\ x &= y \end{aligned}$$

Thus, h is injective.

Exercise 2.10. ⋆

S is uncountable.

Assume by contradiction that there exists a bijection $f : \mathbb{N} \to S$. Then, let $f(n)_i$ denote the ith element in the sequence.

We can construct a new sequence, such that

$$a_n = \begin{cases} 0 & f(n)_n = 1 \\ 1 & f(n)_n = 0 \end{cases}$$

Then, this new sequence will differ from $f(1)$ at the first element, from $f(2)$ at the second element, and so forth. The new sequence is in S because $a_n \in \{0, 1\}$, but is not equal to $f(n)$ for any n. Thus, f is not surjective, leading to a contradiction.

Exercise 2.12. ⋆

(a) $\{(0, 1)\}$

(b) A collection of uncountably many disjoint open intervals does not exist.

Because $\mathbb{Q}$ is dense in $\mathbb{R}$, every nonempty open interval in $\mathbb{R}$ must contain at least one rational number. Because each interval is disjoint, by definition, they may not share any elements, and thus they must not contain the same rational number.

Then, let f be a function mapping these open disjoint intervals to a rational number. f must be injective as at most one interval can contain a given rational. Then, $|\text{set of disjoint open intervals}| \leq |\mathbb{Q}| < |\mathbb{R}|$, thus a collection of uncountably many disjoint open intervals cannot exist.

Exercise 2.13. ⋆

Exercise 2.16. ⋆

Let $A \subseteq \mathbb{N}$.

If A has an upper bound, then let $n \in \mathbb{N}$ be an upper bound such that $\forall a \in A : n \geq a$. Then, $|A| < n$, because there are only n naturals $\leq n$. Then, A must be finite.

Otherwise, if A does not have an upper bound, such that $\forall n : \exists a : (a \in A) \wedge (a > n)$, we construct a bijection $f : \mathbb{N} \to A$, such that $f(n) = a$ such that $a \in A$ and there are exactly n elements in $A \leq a$.

f is injective. Assume that $f(x) = f(y)$. Then,

Exercise 2.22. ⋆